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SERIES SOLUTION OF FREEZING PROBLEM WITH THE FIXED SURFACE RADIATING INTO A MEDIUM OF ARBITRARY VARYING TEMPERATURE*

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Abstract—The series solution of the one-dimensional freezing problem has been found for the case that Newton's law of cooling holds at the fixed boundary. Using a method due to Portnov the position of the progressing phase-change front has been obtained by a series expansion in powers of $\sqrt{t}$. The coefficients up to the power $n = 8$ are given. The formulae have been applied to an example. An estimate for the truncation error as a function of a dimensionless parameter has been obtained.

NOMENCLATURE

H, exterior conductivity [cal/cm^2 s °C];
K, thermal conductivity [cal/cm s °C];
k, thermal diffusivity [cm^2/s];
L, latent heat [cal/g];
T_0, reference temperature ($= 1$°C);
T_s, temperature of solidification [°C];
T_1, $= T_s - 1$ [°C];
t, time [s];
U, ambient temperature [°C];
u, $(U - T_1)/T_0$;
U_n, u_n, coefficients in the power series expansion of the ambient temperature;
V, temperature function in the solid [°C];
v, $= (V - T_1)/T_0$;
v_{1n}, v_{2n}, $= \phi_{1n}(K/H)^n/T_0$; $= \phi_{2n}(K/H)^n/T_0$;
X, location of interface [cm];
X_n, coefficients in the power series expansion of the interface location;
x, space co-ordinate [cm];
y, variable of integration.

Greek symbols

ξ, XH/K;
ξ_n, coefficients in the power-series expansion of ξ;
η, $2k\rho L/KT_0$;
ϑ, $2\sqrt{(kt)}$ [cm];
θ, $[2H\sqrt{(kt)}]/K$;
ρ, density [g/cm^3];
ζ, xH/K;
γ, X/ϑ;
$\Phi_1(\omega)$, $\Phi_2(\omega)$, fictitious temperature functions;
ϕ_{1n}, ϕ_{2n}, coefficients in the power-series expansion of Φ_1 and Φ_2.

INTRODUCTION

THE PROBLEM of heat conduction through a solid in the presence of a change of phase was formulated by Stefan in 1889 [1]. Assuming a water mass to be at the temperature of solidification he found, stipulating the temperature at the surface, the position of the moving boundary (freezing line) using an approximate method. The exact solution for the above case was given by Neumann [2].

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Problems of this type are non-linear because they involve a boundary whose position is not known *a priori*. Therefore one cannot obtain solutions for other boundary conditions by superimposing Neumann solutions.

For this reason a great number of investigators, in particular those dealing with problems which arise in aerospace science have concentrated their efforts on numerical methods [3–6]. An entirely different approach has been taken by Goodman [7] who utilized the heat-balance integral in order to solve problems in heat conduction involving a change of phase.

Using a power series expansion in time, Evans *et al.* [8] have attempted to solve the problem of freezing under the assumption that at the fixed boundary the heat flux is a prescribed function of time.

Recently Stephan [9] has attempted to solve the freezing problem under the assumption of a radiation boundary condition (= Newton's law of cooling) by superimposing Neumann solutions.

In 1962 Portnov [10] suggested a method which made it possible to solve problems in heat conduction involving a change of phase with more involved boundary conditions at the fixed boundary. Jackson [11] has examined the theory of the method in detail and has applied it to various problems in connexion with the melting and the solidification of finite slabs.

In this work the method due to Portnov has been applied to the problem of finding the location of the progressing phase-change front under the assumption that the radiation boundary condition has been imposed at the fixed surface.

FORMULATION

Consider a liquid filling the space from $x = 0$, which we take to be the fixed boundary, to $x = -\infty$. Assuming that the phase-change front has progressed to $x = X(t)$, the one-dimensional differential equation of heat conduction must hold in the solidified material, i.e.

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2} \tag{1}$$

where $V(x, t)$ is the temperature function and k the constant of thermal diffusion. Since we

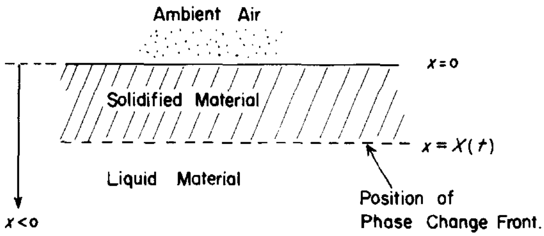

FIG. 1.

assume that the liquid originally filling the space is at the constant temperature of solidification, only one equation of type (1) is required.

At the fixed boundary we impose the radiation boundary condition

$$-K\frac{\partial V}{\partial x} = H[V - U(t)] \tag{2}$$

where H is the exterior conductivity and K the thermal conductivity. In equation (2) we have assumed that the temperature of the ambient air is an arbitrary function of time.

At the moving boundary, the temperature is always at the solidification temperature T_s; therefore

$$V[X(t), t] = T_s. \tag{3}$$

When the liquid solidifies the latent heat is set free; therefore

$$K\frac{\partial V}{\partial x} = \rho L\frac{\mathrm{d}X}{\mathrm{d}t} \qquad x = X(t) \tag{4}$$

where ρ is the density of the liquid and L is the latent heat of the solidifying material.

As an initial condition we stipulate that at the beginning there is no solid part, i.e.

$$X(0) = 0. \tag{5}$$

Equations (1–5) constitute the formulation of the problem which we desire to solve by applying the method of Portnov.

SOLUTION

Portnov obtains a formal solution of equation (1) without satisfying any boundary condition, viz:

$$V(x,t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \left\{ \exp\left[- \frac{X^2(x/X - y)^2}{\vartheta^2} \right] \right\} \Phi_1(Xy) \frac{X}{\vartheta} \, dy + \frac{1}{\sqrt{\pi}} \int_{-\infty}^0 \left\{ \exp\left[- \frac{X^2(x/X - y)^2}{\vartheta^2} \right] \right\} \Phi_2(Xy) \frac{X}{\vartheta} \, dy \tag{6}$$

where $\vartheta = 2(\sqrt{kt})$. The functions $\Phi_1(\omega)$, $\Phi_2(\omega)$ and $X(t)$ are unknown and have to be determined using the boundary and initial conditions (2–5).

Jackson [11] has shown that $V(x, t)$ defined by (6) is, under suitable restrictions on the functions $\Phi_1(\omega)$, $\Phi_2(\omega)$ and $X(t)$, a solution of the heat-conduction equation (1).

In order to determine the three unknown functions $\Phi_1(\omega)$, $\Phi_2(\omega)$ and $X(t)$, one assumes that these functions and the ambient temperature $U(t)$ can be expanded in power series, i.e.

$$\Phi_1(\omega) = \sum_0^\infty \phi_{1n} \omega^n \tag{7a}$$

$$\Phi_2(\omega) = \sum_0^\infty \phi_{2n} \omega^n \tag{7b}$$

$$X(\vartheta) = \sum_1^\infty X_n \vartheta^n \tag{7c}$$

$$U(\vartheta) = \sum_0^\infty U_n \vartheta^n \tag{7d}$$

where expansion (7c) already satisfies the initial condition (5).

Substituting (6) into the boundary condition (2) gives

$$(\sqrt{\pi})\, \vartheta H U(\vartheta) = 2K \left[\int_0^\infty \{\beta \exp(-\beta)^2\} \Phi_1(\beta\vartheta) \, d\beta + \int_{-\infty}^0 \{\beta \exp(-\beta)^2\} \Phi_2(\beta\vartheta) \, d\beta \right] + H\vartheta \left[\int_0^\infty \{\exp(-\beta^2)\} \Phi_1(\beta\vartheta) \, d\beta + \int_{-\infty}^0 \{\exp(-\beta^2)\} \Phi_2(\beta\vartheta) \, d\beta \right] \tag{8}$$

where β is defined by

$$\beta^2 = \frac{X^2(1 - y)^2}{\vartheta^2} . \tag{9}$$

To obtain a relation between the coefficients ϕ_{1n}, ϕ_{2n}, X_n and U_n we substitute the expansions (7) into (8). Letting ϑ tend towards zero, gives

$$\phi_{10} - \phi_{20} = 0. \tag{10}$$

Substituting (6) into the boundary condition (3) gives

$$T_s = (1/\sqrt{\pi}) \left[\int_{-X(\vartheta)/\vartheta}^\infty \{\exp(-\beta)^2\} \Phi_1(\beta\vartheta + X) \, d\beta + \int_{-\infty}^{-X(\vartheta)/\vartheta} \{\exp(-\beta^2)\} \Phi_2(\beta\vartheta + X) \, d\beta \right]. \tag{11}$$

Introducing the expansions (7) into equation (11) and letting ϑ tend towards zero finally gives

$$\phi_{10} = \phi_{20} = T_s . \tag{12}$$

In order to obtain one more equation for determining the first three coefficients ϕ_{10}, ϕ_{20} and X_1, we substitute (6) into the boundary condition (4),

$$2k\rho L\,(\mathrm{d}X/\mathrm{d}\vartheta) = (2K/\sqrt{\pi})\Big[\int_{-X(\vartheta)/\vartheta}^{\infty}\{\beta\exp(-\beta^2)\}\,\Phi_1(\beta\vartheta + X)\,\mathrm{d}\beta + \int_{-\infty}^{-X(\vartheta)/\vartheta}\{\beta\exp(-\beta^2)\}\,\Phi_2(\beta\vartheta + X)\,\mathrm{d}\beta\Big]. \tag{13}$$

On substituting into (13) the expansions (7), we obtain, on letting ϑ tend to zero,

$$2k\rho L X_1 = (K/\sqrt{\pi})\{\phi_{10} - \phi_{20}\}\exp(-X_1^2)$$

or because of (12)

$$X_1 = 0. \tag{14}$$

In order to obtain the coefficients ϕ_{11}, ϕ_{21} and X_2 in the power-series expansions (7), we take the first derivative of (8), (11) and (13) with respect to ϑ. Substituting the expansions (7) into the equations and letting ϑ tend to zero, we obtain after the evaluation of the definite integrals three equations from which the coefficients can be determined.

The first derivative of (8) with respect to ϑ is

$$(\sqrt{\pi})\,H\{U(\vartheta) + \vartheta U^{(1)}(\vartheta)\} = 2K\Big[\int_0^{\infty}\{\beta^2\exp(-\beta^2)\}\,\Phi_1^{(1)}(\beta\vartheta)\,\mathrm{d}\beta + \int_{-\infty}^{0}\{\beta^2\exp(-\beta^2)\}\,\Phi_2^{(1)}(\beta\vartheta)\,\mathrm{d}\beta\Big] + H\Big[\int_0^{\infty}\{\exp(-\beta^2)\}\,\Phi_1(\beta\vartheta)\,\mathrm{d}\beta + \int_{-\infty}^{0}\{\exp(-\beta^2)\}\,\Phi_2(\beta\vartheta)\,\mathrm{d}\beta\Big]$$
$$+ H\vartheta\Big[\int_0^{\infty}\{\beta\exp(-\beta^2)\}\,\Phi_1^{(1)}(\beta\vartheta)\,\mathrm{d}\beta + \int_{-\infty}^{0}\{\beta\exp(-\beta^2)\}\,\Phi_2^{(1)}(\beta\vartheta)\,\mathrm{d}\beta\Big]$$

where $\Phi_1^{(1)}(\beta\vartheta)$ and $\Phi_2^{(1)}(\beta\vartheta)$ are the first derivatives of the functions with respect to their arguments. On substituting the expansions (7) into the last equation we obtain finally

$$\phi_{11} + \phi_{21} = \frac{2H}{K}(U_0 - T_s). \tag{15}$$

The first derivative of (11) with respect to ϑ is

$$0 = (1/\sqrt{\pi})\Big[\int_{-\gamma}^{\infty}\{\exp(-\beta^2)\}(\beta + X^{(1)})\,.\,\Phi_1^{(1)}(\beta\vartheta + X)\,\mathrm{d}\beta + \{\gamma^{(1)}\exp(-\gamma^2)\}\,\Phi_1(0) + \int_{-\infty}^{-\gamma}\{\exp(-\beta^2)\}(\beta + X^{(1)})\,.\,\Phi_2^{(1)}(\beta\vartheta + X)\,\mathrm{d}\beta - \{\gamma^{(1)}\exp(-\gamma^2)\}\,\Phi_2(0)\Big]$$

where $\gamma = X(\vartheta)/\vartheta$. On substituting the expansions (7) into the above equation and letting ϑ tend to zero, we obtain after evaluating the integrals

$$\phi_{11} - \phi_{21} = 0. \tag{16}$$

Performing the same steps as above on (13) gives

$$X_2 = \frac{K\phi_{11}}{4kpL}$$

which by virtue of (15) and (16) may be written as

$$X_2 = \frac{H(U_0 - T_s)}{4kpL} \tag{17}$$

which is always negative since $U_0 < T_s$.

In order to obtain the higher-order coefficients, we continue in the manner described above, that is, we take the higher-order derivatives of equations (8), (11) and (13), let ϑ tend to zero and evaluate the integrals. From the resulting three equations we obtain the unknown coefficients ϕ_{1n}, ϕ_{2n} and X_{n+1} where n is the order of the derivatives.

Since it is desirable to reduce the number of parameters in the coefficients ϕ_{1n}, ϕ_{2n} and X_{n+1} as much as possible, we introduce the following dimensionless quantities:

$$\xi = \frac{H}{K}X; \qquad \theta = 2\frac{H}{K}\sqrt{kt} \tag{18a}$$

$$\zeta = \frac{H}{K}x; \qquad \eta = \frac{2k\rho L}{KT_0} \tag{18b}$$

$$v(\zeta, \theta) = \frac{V(x, t) - T_1}{T_0}; \qquad u(\theta) = \frac{U(t) - T_1}{T_0} \tag{18c}$$

with $T_1 = T_s - 1°C$ and $T_0 = 1°C$. These quantities have been introduced into the coefficients ϕ_{1n}, ϕ_{2n} and X_{n+1}. The resulting dimensionless coefficients v_{1n}, v_{2n} and ξ_{n+1} have been listed in Appendix A.

THE TEMPERATURE OF THE SURFACE

The temperature at the fixed surface, i.e. at $\zeta = 0$, for $\theta \geqslant 0$ is given by the dimensionless form of equation (6)

$$v(0, \theta) = (1/\sqrt{\pi})\left[\int_0^{\infty} \{\exp(-\beta^2)\}\, v_1(\beta\theta)\, d\beta + \int_{-\infty}^{0} \{\exp(-\beta^2)\}\, v_2(\beta\theta)\, d\beta\right]. \tag{19}$$

Substitution of the power-series expansion of $v_1(\beta\theta)$ and $v_2(\beta\theta)$ into the above equation gives after rearranging and evaluating the integrals

$$v(0, \theta) = \frac{1}{\sqrt{\pi}}\left\{(\sqrt{\pi}) + \tfrac{1}{2}(v_{11} - v_{21})\theta + \frac{\sqrt{\pi}}{4}(v_{12} + v_{22})\theta^2 + \tfrac{1}{2}(v_{13} - v_{23})\theta^3 + \frac{3\sqrt{\pi}}{8}(v_{14} + v_{24})\theta^4 + (v_{15} - v_{25})\theta^5 + \frac{15\sqrt{\pi}}{16}(v_{16} + v_{26})\theta^6 + 3(v_{17} - v_{27})\theta^7 + \dots\right\}. \tag{20}$$

Putting $v_{1n} = v_{2n}$ which is the case if the ambient temperature is a function of t rather than $\sqrt{t}$, we obtain

$$v(0, \theta) = 1 + \frac{v_{12}}{2}\theta^2 + \frac{3v_{14}}{4}\theta^4 + \frac{15v_{16}}{8}\theta^6 + \dots \tag{21}$$

where the v_{1n} are the coefficients listed in the Appendix.

APPLICATION TO AN EXAMPLE

In order to apply the derived formulae to an example we assume that the solidifying medium extends infinitely in both horizontal directions as well as infinitely downwards from $x = 0$. Given a specific ambient temperature which must be expandable as a power series in the square root of time we are able to compute the position of the interface as a function of time.

Apart from the coefficients of the power series expansion of the ambient temperature the only dimensionless parameter which enters the formulae of Appendix A is η. Selecting water as a solidifying medium η becomes approximately 350.

Using a digital computer of the type I.B.M. 1620 the formulae of Appendix A were evaluated for four different constant ambient temperatures u_0. Using the coefficients ξ_n, the position of the phase-change front has been calculated from the dimensionless form of the power-series expansion (7c). The result of this computation has been plotted in Fig. 2.

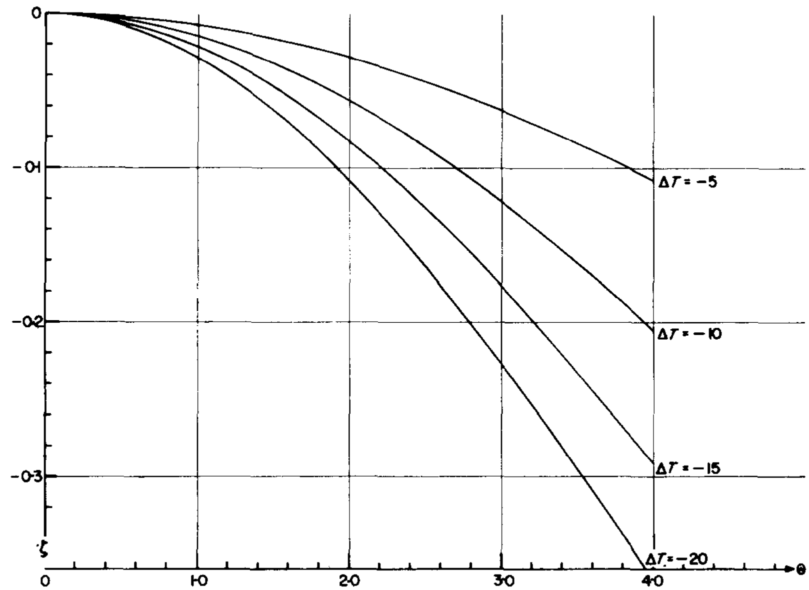

FIG. 2. Thickness of solidified material vs. θ (constant ambient temperature).

DISCUSSION

In Fig. 2 we have plotted the dimensionless location ξ of the phase-change front against the dimensionless variable θ. The four curves in Fig. 2 correspond to four temperature differences, ΔT, between the temperature of solidification and the ambient temperature.

Since the coefficient ξ_1 is identically zero, the phase-change front begins to move with a finite velocity (the velocity is obtained by multiplying the quantity $d\xi/\theta \, d\theta$ by $2kH/K$). As one may expect the rate of growth is greatest at the very beginning and it falls off as the thickness increases. Taking a difference of 20 degC between the ambient and solidification temperature the quantity $d\xi/\theta \, d\theta$ becomes 0·0571. Assuming an emissivity of the ice surface of 0·9 the constant H takes a value of $7{\cdot}71 \times 10^{-5}$ (c.g.s.) where the ambient temperature was taken to 250°K [12]. With that value the velocity of the phase-change front at the beginning becomes 1·65 cm/day which is obviously too low. Since the relationship between H and the rate of growth is linear, one can easily adjust H in order to make the rate of growth at the beginning fit the experimental results. Therefore, in our case, if we assume that the rate of growth at a temperature difference of 20 degC is initially five times as large as quoted above, one finds that $H = 3{\cdot}86 \times 10^{-4}$ (c.g.s.).

Using the method of the heat-balance integral Goodman [7] has obtained an approximate solution for the case which has been studied here. Assuming the temperature distribution within the solid to be parabolic rather than linear [7], he obtains the time as a function of the position of the phase-change front, i.e.

$$\begin{aligned} T = \frac{1}{12\beta}\Bigg[&\{(1 + 2\beta) + (2 + \beta)S\}\{1 + \beta S(2 + S)\}^{\frac{1}{2}} \\ &- \frac{2(\beta - 1)}{\sqrt{\beta}}\ln\left\{\frac{[1 + \beta S(2 + S)]^{\frac{1}{2}} + [(1 + S)\beta]^{\frac{1}{2}}}{1 + \sqrt{\beta}}\right\} \\ &- 4\beta(\beta - 1)\ln\left\{\frac{- 1 + \beta(2 + S) + [1 + \beta S(2 + S)]^{\frac{1}{2}}}{2\beta}\right\} \\ &+ (\beta^2 + 5\beta)\frac{S^2}{2} + 2(\beta^2 + 4\beta - 2)S - (1 + 2\beta)\Bigg] \end{aligned} \tag{22}$$

where the quantities S, β and T are connected with the variables used in this paper as follows:

$$T = \frac{\theta^2|u_0 - 1|}{2\eta}, \qquad S = -\xi, \qquad \beta = 1 + \frac{4|u_0 - 1|}{\eta}. \tag{23}$$

Since it would not be feasible to invert equation (22) we have substituted the location of the phase-change front as obtained from the above developed method into the right-hand side of equation (22).

In Table 1 we have shown the numerical values of ξ in connexion with the corresponding θ. Substituting ξ into equations (22) and (23) one obtains θ_g. Taking the case for which $\eta = 350$ and $\Delta T = -10$ one notices the agreement between the values of θ and θ_g.

Table 1

θ	ξ	θ_g
0·5	$0{\cdot}3565 \times 10^{-2}$	0·501
1·0	$0{\cdot}1418 \times 10^{-1}$	1·002
1·5	$0{\cdot}3161 \times 10^{-1}$	1·503
2·0	$0{\cdot}5562 \times 10^{-1}$	2·004
2·5	$0{\cdot}8543 \times 10^{-1}$	2·505
3·0	0·1208	3·005
3·5	0·1611	3·504
4·0	0·2056	4·004
4·5	0·2534	4·503
5·0	0·3028	5·002

One of the features of the method was that the position of the phase-change front has been expanded in a power series in powers of θ with the coefficients ξ_n being of alternating sign. This fact enables us to estimate the error which we commit if terms with the power higher than eight in θ are left out. Putting $\theta = 1$ we have plotted in Fig. 3 the ratio of $\xi_8\theta^8$ to ξ as a function of η for a number of temperature differences, ΔT. The truncation error in this case is less than 0·002 per cent.

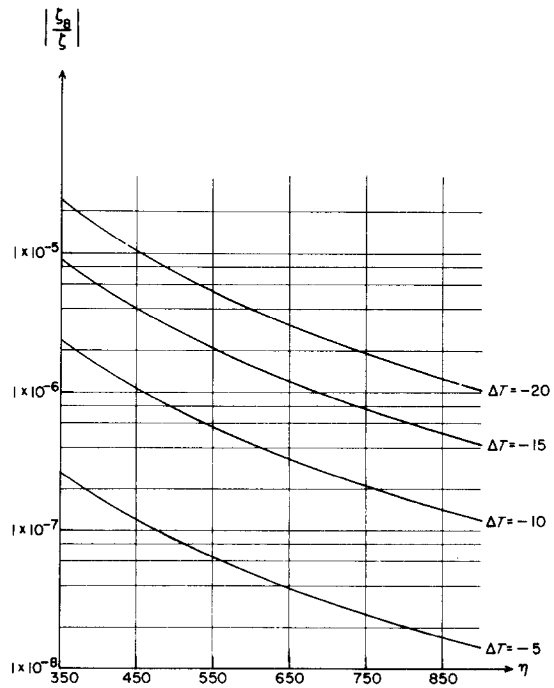

FIG. 3. Relative truncation error for $\theta = 1$ vs. η.

Increasing θ to $\theta = 4$ preserves the form of the curves but the truncation error now increases to less than 1·2 per cent.

Remembering that the parameter η which for materials other than water (i.e. steel, aluminium) assumes values in the range 800–1000, one concludes that the analysis is not confined to materials with high latent heat, and that the eight terms represent a good approximation of the converging power series for ξ.

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APPENDIX A

$$v_{10} = v_{20} = 1$$

$$\xi_1 = 0$$

$$v_{11} + v_{21} = 2(u_0 - 1)$$

$$v_{11} - v_{21} = 0$$

$$\xi_2 = \frac{v_{11}}{2\eta}$$

$$v_{12} - v_{22} = (\sqrt{\pi})\, u_1$$

$$v_{12} + v_{22} = -\,2\xi_2(v_{11} + v_{21})$$

$$\xi_3 = \frac{v_{12} - v_{22}}{3\eta\sqrt{\pi}}$$

$$v_{13} + v_{23} = \left\{\frac{4u_2 - (v_{12} + v_{22})}{3}\right\}$$

$$v_{13} - v_{23} = -\,\{\xi_3(\sqrt{\pi})(v_{11} + v_{21}) + 2\xi_2(v_{12} - v_{22})\}$$

$$\xi_4 = (1/12\eta)\,\{3\xi_2(v_{12} + v_{22}) + 2{\cdot}25(v_{13} + v_{23})\}$$

$$v_{14} - v_{24} = \left\{\frac{2(\sqrt{\pi})\, u_3 - (v_{13} - v_{23})}{4}\right\}$$

$$v_{14} + v_{24} = -\frac{4\xi_4(\sqrt{\pi})(v_{11} + v_{21}) + 8\xi_3(v_{12} - v_{22}) + 6\xi_2(\sqrt{\pi})(v_{13} + v_{23}) + 4\xi_2^2(\sqrt{\pi})(v_{12} + v_{22})}{3\sqrt{\pi}}$$

$$\xi_5 = \frac{\xi_3(\sqrt{\pi})(v_{12} + v_{22}) + 3\xi_2(v_{13} - v_{23}) + \xi_2^2(v_{12} - v_{22}) + 2(v_{14} - v_{24})}{5(\sqrt{\pi})\,\eta}$$

$$v_{15} + v_{25} = \left\{\frac{8u_4 - 3(v_{14} + v_{24})}{15}\right\}$$

$$v_{15} - v_{25} = -\frac{1}{12}\{6\xi_5(\sqrt{\pi})(v_{11} + v_{21}) + 12\xi_4(v_{12} - v_{22}) + 9\xi_3(\sqrt{\pi})(v_{13} + v_{23}) + 24\xi_2(v_{14} - v_{24}) + 12\xi_2\xi_3(\sqrt{\pi})(v_{12} + v_{22}) + 18\xi_2^2(v_{13} - v_{23}) + 4\xi_2^3(v_{12} - v_{22})\}$$

$$\xi_6 = \frac{1}{48(\sqrt{\pi})\eta}\{8\xi_4(\sqrt{\pi})(v_{12} + v_{22}) + 24\xi_3(v_{13} - v_{23}) + 24\xi_2(\sqrt{\pi})(v_{14} + v_{24}) + 16\xi_2\xi_3(v_{12} - v_{22}) + 12\xi_2^2(\sqrt{\pi})(v_{13} + v_{23}) + 15(\sqrt{\pi})(v_{15} + v_{25})\}$$

$$v_{16} - v_{26} = \left\{\frac{u_5(\sqrt{\pi}) - (v_{15} - v_{25})}{6}\right\}$$

$$v_{16} + v_{26} = -\frac{1}{15\sqrt{\pi}}\{8\xi_6(\sqrt{\pi})(v_{11} + v_{21}) + 16\xi_5(v_{12} - v_{22}) + 12\xi_4(\sqrt{\pi})(v_{13} + v_{23}) + 32\xi_3(v_{14} - v_{24}) + 30\xi_2(\sqrt{\pi})(v_{15} + v_{25}) + 16\xi_2\xi_4(\sqrt{\pi})(v_{12} + v_{22}) + 48\xi_2\xi_3(v_{13} - v_{23}) + 24\xi_2^2(\sqrt{\pi})(v_{14} + v_{24}) + 8\xi_3^2(\sqrt{\pi})(v_{12} + v_{22}) + 8\xi_2^3(\sqrt{\pi})(v_{13} + v_{23}) + 8\xi_2^2\xi_3(v_{12} - v_{22})\}$$

$$\xi_7 = \frac{1}{42(\sqrt{\pi})\eta}\{6\xi_5(\sqrt{\pi})(v_{12} + v_{22}) + 18\xi_4(v_{13} - v_{23}) + 18\xi_3(\sqrt{\pi})(v_{14} + v_{24}) + 60\xi_2(v_{15} - v_{25}) + 12\xi_2\xi_4(v_{12} - v_{22}) + 18\xi_2\xi_3(\sqrt{\pi})(v_{13} + v_{23}) + 36\xi_2^2(v_{14} - v_{24}) + 6\xi_3^2(v_{12} - v_{22}) + 6\xi_2^3(v_{13} - v_{23}) + 36(v_{16} - v_{26}) - \xi_2^4(v_{12} - v_{22})\}$$

$$v_{17} + v_{27} = \left\{\frac{16u_6 - 15(v_{16} + v_{26})}{105}\right\}$$

$$v_{17} - v_{27} = -\frac{1}{360}\{60\xi_7(\sqrt{\pi})(v_{11} + v_{21}) + 120\xi_6(v_{12} - v_{22}) + 90\xi_5(\sqrt{\pi})(v_{13} + v_{23}) + 240\xi_4(v_{14} - v_{24}) + 225\xi_3(\sqrt{\pi})(v_{15} + v_{25}) + 720\xi_2(v_{16} - v_{26}) + 120\xi_2\xi_5(\sqrt{\pi})(v_{12} + v_{22}) + 360\xi_2\xi_4(v_{13} - v_{23}) + 360\xi_2\xi_3(\sqrt{\pi})(v_{14} + v_{24}) + 600\xi_2(v_{15} - v_{25}) + 120\xi_3\xi_4(\sqrt{\pi})(v_{12} + v_{22}) + 180\xi_2^2\xi_3(\sqrt{\pi})(v_{13} + v_{23}) + 180\xi_3^2(v_{13} - v_{23}) + 240\xi_2^3(v_{14} - v_{24}) + 120\xi_2\xi_3^2(v_{12} - v_{22}) + 120\xi_2^2\xi_4(v_{12} - v_{22}) - 4\xi_2^5(v_{12} - v_{22}) + 30\xi_2^4(v_{13} - v_{23})\}$$

$$\xi_8 = \frac{1}{1344(\sqrt{\pi})\eta}\{168\xi_6(\sqrt{\pi})(v_{12} + v_{22}) + 504\xi_5(v_{13} - v_{23}) + 504\xi_4(\sqrt{\pi})(v_{14} + v_{24}) + 1680\xi_3(v_{15} - v_{25}) + 1890\xi_2(\sqrt{\pi})(v_{16} + v_{26}) + 336\xi_2\xi_5(v_{12} - v_{22}) + 504\xi_2\xi_4(\sqrt{\pi})(v_{13} + v_{23}) + 2016\xi_2\xi_3(v_{14} - v_{24}) + 1260\xi_2^2(\sqrt{\pi})(v_{15} + v_{25}) + 336\xi_3\xi_4(v_{12} - v_{22}) + 504\xi_2^2\xi_3(v_{13} - v_{23}) + 252\xi_3^2(\sqrt{\pi})(v_{13} + v_{23}) + 336\xi_2^3(\sqrt{\pi})(v_{14} + v_{24}) + 1102{\cdot}5(\sqrt{\pi})(v_{17} + v_{27}) - 112\xi_2^3\xi_3(v_{12} - v_{22}) - 2{\cdot}4\xi_2^4(v_{13} - v_{23})\}$$

Résumé— La solution en série du problème de la congélation unidimensionnelle a été trouvée dans le cas où la loi de refroidissement de Newton est valable sur la frontière fixe. En utilisant une méthode due à Portnov, on a obtenu la position mobile du front de changement de phase sous la forme de développement en série de puissances de $\sqrt{t}$. Les coefficients jusqu'à la puissance $n = 8$ sont donnés. Les formules ont été appliquées à un exemple. Une estimation de l'erreur de troncature en fonction d'un paramètre sans dimensions a été obtenue.

Zusammenfassung—Die Reihenlösung für das eindimensionale Gefrierproblem liess sich für den Fall finden, dass Newton's Abkühlungsgesetz für die feste Berandung gilt. Nach einer Methode von Portnov kann der Ort der fortschreitenden Front der Phasenänderung durch eine Reihenentwicklung nach Potenzen von $\sqrt{t}$ erhalten werden. Die Koeffizienten bis zur Potenz $n = 8$ sind angegeben. Die Formeln wurden für ein Beispiel angewandt. Eine Abschätzung für den Abbruchfehler wurde als Funktion eines dimensionslosen Parameters erhalten.

Аннотация—Получено решение в виде степенного ряда одномерной задачи для случая, когда закон Ньютона для охлаждения справедлив на неподвижной границе. Распространение фронта изменения фазы получено в виде степенного ряда по степеням $\sqrt{t}$ по методу Портнова. Приводятся коэффициенты членов со степенью до $n = 8$. Применение формул проиллюстрировано примерами. Получена оценка ошибки вследствии отбрасывания членов в зависимости от безразмерного параметра.